Lipschitz-free spaces over unit spheres and the Metric Approximation Property

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Lipschitz-free Banach spaces

Definition

Let (M, d) be a metric space having distinguished point x_0. Define $\mathrm{Lip}_0(M)$ to be the space of all Lipschitz functions $f : M \to \mathbb{R}$ that vanish at x_0, with norm

$$\|f\| := \mathrm{Lip}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x,y)} \ : \ x, y \in M,\, x \neq y\right\}.$$

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$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|}(\delta_x)_{x\in M} \subseteq \mathrm{Lip}_0(M)^*.$$

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Fact

$\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$.

Approximation properties

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A Banach space X has

1. the **approximation property (AP)** if, given $K \subseteq X$ compact and $\varepsilon > 0$, there is a finite-rank operator T on X such that $\|Tx - x\| \leqslant \varepsilon$ for all $x \in K$;

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Proposition

X has the λ-BAP if and only if $I \in \lambda\overline{B}^{\mathrm{SOT}}$, where B is the unit ball of the space of finite-rank operators on X.

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In particular, if there is a sequence $(F_n)_{n=1}^{\infty}$ of finite-rank operators satisfying $\|F_n\| \to 1$ and $\|F_n x - x\| \to 0$ for all x, then X has the MAP.

Free spaces and approximation properties

MAP Theorems

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5. M is a compact group with a left-invariant metric (Doucha, Kaufmann 20);
6. M is a compact subset of $\mathbb{R}^N$ that is purely 1-unrectifiable (Aliaga, Gartland, Petitjean, Procházka 22).

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Question (Godefroy 15)

If $M \subseteq (\mathbb{R}^N, \|\cdot\|)$ does $\mathcal{F}(M)$ have the MAP or λ-BAP (independent of N)?

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1. There is a compact convex subset M of a Banach space X, such that $\mathcal{F}(M)$ fails the AP (Godefroy, Ozawa 14).

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1. There is a compact convex subset M of a Banach space X, such that $\mathcal{F}(M)$ fails the AP (Godefroy, Ozawa 14).
2. There is a compact space M homeomorphic to the Cantor set, such that $\mathcal{F}(M)$ fails the AP (Hájek, Lancien, Pernecká 16)

The MAP and $\mathcal{F}(M)$, where $M \subseteq \mathbb{R}^N$

Consider $M \subseteq (\mathbb{R}^N, |||\cdot|||)$ with distinguished point x_0. As $\mathcal{F}(M) \equiv \mathcal{F}(\overline{M})$, we assume M is closed.

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Theorem A (Pernecká, Smith 15)

Let M be compact and have the property that, given $\varepsilon > 0$, there exists $\hat{M} \subseteq \mathbb{R}^N$ and an 'almost retraction' $\Psi : \hat{M} \to M$, such that

$$M \subseteq \operatorname{int}(\hat{M}), \quad |||\cdot|||\text{-}\mathrm{Lip}(\Psi) \leqslant 1+\varepsilon \quad \text{and} \quad |||x - \Psi(x)||| \leqslant \varepsilon \text{ for all } x \in \hat{M}.$$

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- Theorem A applies to '**locally downwards closed**' compact sets M. This means that, locally, the boundary of M is (under a change of coordinates) the graph of a function from $\mathbb{R}^{N-1}$ to $\mathbb{R}$.
- By Theorem A, if M is compact and convex then $\mathcal{F}(M)$ has the MAP with respect to any norm.

The failure of Theorem A for unit spheres

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Then for all small enough $\varepsilon > 0$, by translation, there exists open $U \supseteq C$ and $\Psi : U \to D$ such that

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Given $\varepsilon > 0$, Ψ and U, by Rademacher's Theorem $\Psi'(x)$ exists a.a. $x \in U$. We have $|||\Psi'(x)||| \leqslant 1 + \varepsilon$, and $\operatorname{ran} \Psi'(x) \subseteq X$ because $D \subseteq X$.

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Let $a > 0$ such that $[0,1]^2 \times [-a,a] \subseteq U$ and define linear $T : \mathbb{R}^3 \to X$ by

$$T = \frac{1}{2a} \int_{-a}^{a} \int_0^1 \int_0^1 \Psi'(x_1, x_2, x_3)\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3.$$

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We have $|||T||| \leqslant 1 + \varepsilon$ and, using the property $|||x - \Psi(x)||| \leqslant \varepsilon$ for all $x \in U$, it can be shown that

$$|||T(1,0,0) - (1,0,0)||| \, , \; |||T(0,1,0) - (0,1,0)||| \;\leqslant\; 2\varepsilon.$$

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Hence for every $n \in \mathbb{N}$ there exists a linear map $T_n : \mathbb{R}^3 \to X$ such that

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By compactness, there exists $T : \mathbb{R}^3 \to X$ such that $|||T||| = 1$ and $T\restriction_X$ is the identity on X, but this contradicts the assumption that X is not 1-complemented with respect to $|||\cdot|||$.

The MAP and unit spheres

Theorem B (Talimdjioski, Smith 22)

Let $\|\cdot\|$ and $|||\cdot|||$ be norms on $\mathbb{R}^N$ with $\|\cdot\|$ C^1-smooth, and let $S = S_{(\mathbb{R}^N, \|\cdot\|)}$ (with arbitrary distinguished point x_0). Then $\mathcal{F}(S, |||\cdot|||)$ has the MAP.

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1. Construct mollifier operators $S_n : \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(S)$, $n \in \mathbb{N}$, such that

$$\|S_n(f) - f\|_\infty \leqslant \frac{1}{n} \quad \text{and} \quad |||\cdot|||\text{-}\mathrm{Lip}(S_n(f)) \leqslant 1 + \frac{1}{n}$$

   whenever $|||\cdot|||$-$\mathrm{Lip}(f) \leqslant 1$.

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3. Construct finite-rank operators $P_{n,i} : \mathrm{Lip}_0(S) \to C(\overline{U_i})$, $i \leqslant k$, such that
$$\left\| P_{n,i}(S_n(f)) - S_n(f)\restriction_{\overline{U_i}} \right\|_\infty, \; |||\cdot|||\text{-}\mathrm{Lip}(P_{n,i}(S_n(f)) - S_n(f)\restriction_{\overline{U_i}}) \leqslant \frac{1}{n},$$
whenever $|||\cdot|||$-$\mathrm{Lip}(f) \leqslant 1$, and $P_{n,i}(S_n(f))(x_0) = 0$ whenever $x_0 \in U_i$.

The MAP and unit spheres

Theorem B (Talimdjioski, Smith 22)

Let $\|\cdot\|$ and $|||\cdot|||$ be norms on $\mathbb{R}^N$ with $\|\cdot\|$ C^1-smooth, and let $S = S_{(\mathbb{R}^N,|||\cdot|||)}$ (with arbitrary distinguished point x_0). Then $\mathcal{F}(S, |||\cdot|||)$ has the MAP.

To prove this we work mostly in $\mathrm{Lip}_0(S)$. Outline of the proof:

4. Define finite-rank operators $Q_n : \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(S)$ by

$$Q_n(f)(x) = \sum_{i=1}^{k} \alpha_i(x) P_{n,i}(S_n(f))(x), \qquad x \in S.$$

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$$\|Q_n(f) - f\|_\infty \leqslant \frac{1}{n} \quad \text{and} \quad |||\cdot|||\text{-}\mathrm{Lip}(Q_n(f)) \leqslant 1 + \frac{H}{n}$$

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6. The Q_n are dual operators; $\mathcal{F}(S, |||\cdot|||)$ has the MAP by virtue of the corresponding predual operators.

Step 1: the mollifier operators S_n

Fix $\psi : \mathbb{R}^N \setminus \{0\} \to S$ by $\psi(x) = x/\|x\|$. The next lemma follows because $\|\cdot\|$ is C^1-smooth (note $\|\cdot\|$-$\mathrm{Lip}(\psi) > 1$ in general).

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Lemma

Given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\psi(x+z) - \psi(y+z) - (x-y)\| \leqslant \varepsilon\|x-y\|$$

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Given $s \in (0, \frac{1}{2})$ and $f \in \mathrm{Lip}_0(S)$, define the C^∞-smooth map $\hat{f}_s : \mathbb{R}^N \to \mathbb{R}$ by

$$\hat{f}_s(x) = \int_{\mathbb{R}^N} \eta_s(z) f(\psi(x+z))\,\mathrm{d}z$$

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Given $n \in \mathbb{N}$, define S_n on $\mathcal{F}(S)$ by $S_n(f)(x) = \hat{f}_{s_n}(x) - \hat{f}_{s_n}(x_0)$, $x \in S$. Here, s_n is chosen (using the above lemma) so that

$$\|S_n(f) - f\|_\infty \leqslant \frac{1}{n} \quad \text{and} \quad |||\cdot|||\text{-Lip}(S_n(f)) \leqslant 1 + \frac{1}{n}.$$

Steps 2 and 3: the partition of unity and the $P_{n,i}$

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This function is C^1-smooth on $\mathbb{R}^{N-1}$, being a composition of C^1-smooth and C^∞-smooth maps.

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We do this for every $x \in S$ (independently of n). This yields an open cover of S, from which we extract the partition of unity and the $P_{n,i}$.